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Stabilization of the motions of non-autonomous mechanical systems☆

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ABSTRACT

The problem of stabilizing the motions of mechanical systems that can be described by non-autonomous systems of ordinary differential equations is considered. The sufficient conditions for stabilizing of the motions of mechanical systems with assigned forces due to forces of another structure are obtained by constructing a vector Lyapunov function and a reference system. Examples of the solution of the problems of stabilizing the rotational motion of an axisymmetric satellite in an elliptic orbit, a non-tumbling gyro horizon, etc. are considered ©2009

The problem of the stability of the motion of a mechanical system as a function of the structure of the forces acting on it is one of the basic problems in the theory of the stability of motion. It has been investigated for non-autonomous mechanical systems in numerous studies (Refs 1–6 etc.). An approach based on transformation of the state vector of the system that enables non-conservative positional forces to be eliminated from the equations of motion under certain additional conditions has been developed.[3–5] The conditions for the motion of autonomous mechanical systems to be stable under the action of dissipative, gyroscopic, potential and non-conservative positional forces have been obtained using a Lyapunov function of quadratic form.[7–9] Several theorems regarding the exponential stability and stabilization of the motions of non-autonomous mechanical systems with non-conservative forces have also been obtained by using a Lyapunov function of quadratic form.[6]

An approach based on constructing a vector Lyapunov function with components of the vector norm type and a reference system and using the theory of limit functions[10–12] is employed to solve the problem of stabilizing the motions of non-autonomous mechanical systems. Unlike the classical comparison method,[1] this approach enables the requirements improved on the reference system to be relaxed and thereby enables us to supplement the results of other investigators, obtained using a scalar Lyapunov function of quadratic form.

1. The sufficient conditions for stabilizing of the motions of mechanical systems with specified gyroscopic forces

Suppose the equations of the perturbed motion of a mechanical system have the form

$$\ddot{\mathbf{q}} + \mathbf{G}(t)\dot{\mathbf{q}} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{1.1}$$

where $t \geq 0$, $\mathbf{q} \in R^n$ is the vector of the generalized coordinates, the skew-symmetrical matrix $\mathbf{G}(t) \in R^{n\times n}$ with continuous bounded elements describes the gyroscopic forces acting on the system, $\det \mathbf{G}(t) \neq 0, \forall t \geq 0$, $\mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ is a function that contains $\mathbf{q}$ and $\dot{\mathbf{q}}$ in an order higher than the first order, and $\mathbf{Q}(t, \mathbf{0}, \mathbf{0}) = \mathbf{0}$. For system (1.1) and henceforth we will assume that the function $\mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ is continuous and bounded and that it satisfies the Lipsehitz condition with respect to $\mathbf{q}$ and $\dot{\mathbf{q}}$ for every $\gamma = \text{const} > 0$ in the region $\{t \geq 0, |\mathbf{q}| \leq \gamma, |\dot{\mathbf{q}}| \leq \gamma\}$, where $|\cdot|$ denotes a certain norm in the space R^n. Then system (1.1) will be precompact.[10] This means that for any sequence $t_l \to +\infty$ there is a subsequence $t_k \to +\infty$ such that the following convergences occur

$$\mathbf{G}^*(t) = \lim_{t_k \to +\infty} \mathbf{G}(t + t_k), \quad \mathbf{Q}^*(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{d}{dt}\lim_{t_k \to +\infty}\int_0^t \mathbf{Q}(t_k + \tau, \mathbf{q}, \dot{\mathbf{q}})d\tau$$

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The asterisk denotes a function that is a limit function for the original function. The corresponding convergences are uniform for

$$t \in [0, T], \quad (t, \mathbf{q}, \dot{\mathbf{q}}) \in [0, T] \times \{\mathbf{q} : |\mathbf{q}| \le \gamma\} \times \{\dot{\mathbf{q}} : |\dot{\mathbf{q}}| \le \gamma\}$$

where $T = \text{const} > 0$ is an arbitrary number. Thus, for system (1.1) we can construct the family of limit equations

$$\ddot{\mathbf{q}} + \mathbf{G}^*(t)\dot{\mathbf{q}} = \mathbf{Q}^*(t, \mathbf{q}, \dot{\mathbf{q}})$$

We wish to solve the problem of stabilizing the unperturbed motion

$$\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0} \tag{1.2}$$

of system (1.1) up to uniform asymptotic stability by the addition of dissipative and non-conservative forces.

Theorem 1. *In the case of an even number of coordinates $n = 2k$, the unperturbed motion (1.2) of system (1.1) will be stabilized up to uniform asymptotic stability regardless of the form of the non-linear forces by the addition of dissipative forces with diagonal matrix $b(t)\mathbf{E}$ ($b(t)$ is a continuous bounded function, and $\mathbf{E} \in R^{n\times n}$ is the unit matrix) and non-conservative positional forces with matrix $\mathbf{P}(t)$ such that*

$$b(t) \ge b_0 = \text{const} > 0, \quad |\det\mathbf{P}(t)| \ge p_0 = \text{const} > 0$$

$$\mathbf{G}(t) = \mu(t)\mathbf{P}(t), \quad 0 < \text{const} = \mu_0 \le \mu(t) \le \mu_1, \quad \mu(t) \in C^1$$

$$\dot{\mu}(t) \le b(t)\mu(t) - \delta, \quad \delta = \text{const} > 1$$

Proof. After the dissipative and non-conservative forces are added, linearized system (1.1) takes the form

$$\mathcal{L}(\mathbf{q}) = \mathbf{0}; \quad \mathcal{L}(\mathbf{q}) = \ddot{\mathbf{q}} + (b(t)\mathbf{E} + \mathbf{G}(t))\dot{\mathbf{q}} + \mathbf{P}(t)\mathbf{q} \tag{1.3}$$

Let $\mu(t)$ be a certain continuously differentiable bounded function such that

$$\mu(t) \ge \mu_0 = \text{const} > 0, \quad \forall t \ge 0$$

In Eq. (1.3) we make the replacement of variables

$$\mathbf{x}_1 = \mathbf{q}, \quad \mathbf{x}_2 = \mathbf{q} + \mu(t)\dot{\mathbf{q}}$$

We then obtain the system

$$\dot{\mathbf{x}}_1 = -\frac{1}{\mu(t)}\mathbf{x}_1 + \frac{1}{\mu(t)}\mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = (B(t)\mathbf{E} + \mathbf{G}(t) - \mu(t)\mathbf{P}(t))\mathbf{x}_1 - (B(t)\mathbf{E} + \mathbf{G}(t))\mathbf{x}_2 \tag{1.4}$$

where

$$B(t) = b(t) - \frac{1}{\mu(t)} - \frac{\dot{\mu}(t)}{\mu(t)}$$

For system (1.4) we take the vector Lyapunov function in the form

$$\mathbf{V} = (|\mathbf{x}_1|, |\mathbf{x}_2|)^T$$

where $|\cdot|$ denotes a Euclidean norm in the space R^n. Hence we obtain the reference system

$$\dot{u}_1 = -\frac{1}{\mu(t)}u_1 + \frac{1}{\mu(t)}u_2, \quad \dot{u}_2 = |B(t)|u_1 - B(t)u_2 \tag{1.5}$$

According to the conditions of the theorem, this reference system is uniformly stable. It follows from the precompact nature of system (1.5) that there is a family of limit reference systems[11] that differ from system (1.5) by the replacement of the functions $\mu(t)$ and $b(t)$ by the limit functions $\mu^*(t)$ and $b^*(t)$. It is easy to see here that the solutions $(u_1^*(t), u_2^*(t))$ of the limit reference system satisfy the relation[11]

$$\max\{u_1^*(t), u_2^*(t)\} = \text{const} \ge 0, \quad \forall t \ge 0$$

Then, on the basis of the principle of quasi-invariance of the positive limit set of the perturbed motion,[11] it can be stated that if the set

$$\{\max\{V_1, V_2\} = \text{const} > 0\} \tag{1.6}$$

does not contain solutions of a system that is a limit system for (1.4), the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$ of system (1.4) will be uniformly asymptotically stable according to the theorem of asymptotic stability.[11] This fact is proved as follows: on the set (1.6) the solutions of the limit system for (1.4) satisfy the relation[12]

$$\mathbf{P}^*(t)\mathbf{x}_1^* \equiv \mathbf{0}, \quad \forall t \ge 0 \tag{1.7}$$

which, by virtue of the non-degeneracy of the matrix $\mathbf{P}^*(t)$, implies the condition $\mathbf{x}_1^*(t) \equiv \mathbf{0}$, i.e., the solutions of the limit system for (1.4) do not belong to set (1.6). Therefore, the unperturbed motion (1.2) of system (1.1) will be stabilized up to uniform asymptotic stability.

Corollary 1. *In the case of an even number of coordinates $n=2k$, the unperturbed motion (1.2) of system (1.1) will be stabilized up to uniform asymptotic stability regardless of the form of the non-linear forces by adding dissipative forces with diagonal matrix $b(t)\mathbf{E}$ ($b(t)$ is a continuous bounded function and $\mathbf{E}\in R^{n\times n}$ is the unit matrix) and non-conservative positional forces with matrix $\mathbf{P}(t)$ which are such that the following inequalities hold for all $t\geq 0$*

$$b(t)\geq b_0 = \text{const}>0,\quad \mathbf{G}(t) = \mu\mathbf{P}(t),\quad \mu = \text{const}>0,\quad b_0\mu>1$$

$$|\det\mathbf{P}(t)|\geq p_0 = \text{const}>0$$

Remark 1. If the matrix of the the gyroscopic forces in Eq. (1.1) is a function of t and $\mathbf{q}$ and $\mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}})=\mathbf{0}$, then under the conditions of Theorem 1, the function $\mu=\mu(t,\mathbf{q})$ will satisfy the inequality

$$\frac{\partial\mu}{\partial t}+\left(\frac{\partial\mu}{\partial\mathbf{q}}\right)^T\dot{\mathbf{q}}\leq b(t)\mu(t,\mathbf{q})-1,\quad \forall\mathbf{q},\dot{\mathbf{q}}\in R^n\colon |\mathbf{q}|<\varepsilon,\quad |\dot{\mathbf{q}}|<\varepsilon$$

The number $\varepsilon>0$ specifies the region of initial perturbations.

Example 1. Consider the problem of the stabilizing of the steady motion of a balanced gyroscope in gimbals using external moments.

The equations of motion of a gyroscope in gimbals, taking into account the mass of the suspension, have the form[13]

$$(A_0-C_0\sin^2\beta)\ddot{\alpha}-2C_0\cos\beta\sin\beta\dot{\alpha}\dot{\beta}+H\cos\beta\dot{\beta} = M_\alpha$$

$$B_0\ddot{\beta}+C_0\cos\beta\sin\beta\dot{\alpha}^2-H\cos\beta\dot{\alpha} = M_\beta$$

$$A_0 = A+A_1+A_2,\quad C_0 = A+A_2-C_2,\quad B_0 = A+A_2$$

where A_1 is the moment of inertia of the outer gimbal about the axix of rotation, $A_2=B_2$ and C_2 are the moments of inertia of the frame, A is the equatorial moment of inertia of the rotor, M_α and M_β are the corresponding moments of the external forces, α and β are the angles of rotation of the outer gimbal and the frame, respectively, and $H=C(\dot{\varphi}+\dot{\alpha}\sin\beta)$ is the cyclic constant (C is the polar moment of inertia of the rotor).

We will assume that $C_0=0$. Let the external moments acting on the system be

$$M_\alpha = -dA_0\dot{\alpha}-f\beta,\quad M_\beta = -dB_0\dot{\beta}+f\alpha$$

where $d>0$ and $f>0$ are constants. Then, introducing the notation

$$\tilde{\mathbf{x}} = (\tilde{x}_1,\tilde{x}_2)^T,\quad \tilde{x}_1 = \sqrt{A_0}\alpha,\quad \tilde{x}_2 = \sqrt{B_0}\beta,\quad \mathbf{J} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

we can write the system of equations of motion in vector form

$$\ddot{\tilde{\mathbf{x}}}+d\dot{\tilde{\mathbf{x}}}+\frac{H\cos\tilde{x}_2}{\sqrt{A_0B_0}}\mathbf{J}\dot{\tilde{\mathbf{x}}}+\frac{f}{\sqrt{A_0B_0}}\mathbf{J}\tilde{\mathbf{x}} = 0 \tag{1.8}$$

Then we have

$$\mu(t) = \frac{H\cos\tilde{x}_2(t)}{f}$$

Hence, by Theorem 1, we obtain the following condition for stability of the steady motion of a gyroscope

$$\frac{2\varepsilon\sin\varepsilon}{\cos\varepsilon}\leq\frac{dH\cos\varepsilon}{f}-1$$

where $\varepsilon>0$ specifies the region of initial perturbations.

2. The sufficient conditions for stability of the motions of mechanical systems with assigned non-conservative forces

Let the equations of the perturbed motion of a mechanical system have the form

$$\ddot{\mathbf{q}}+\mathbf{P}(t)\mathbf{q} = \mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}}) \tag{2.1}$$

where $\mathbf{q}\in R^n$ is the vector of the generalized coordinates, the skew-symmetric matrix $\mathbf{P}(t)\in R^{n\times n}$ with continuous bounded elements describes the non-conservative positional forces acting on the system, $\mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}})$ is a function that contains $\mathbf{q}$ and $\dot{\mathbf{q}}$ in an order higher than the first order, and $\mathbf{Q}(t,\mathbf{0},\mathbf{0})=\mathbf{0}$. We will assume that the matrix $\mathbf{P}(t)$ is non-degenerate, that $|\det\mathbf{P}(t)|\geq p_0=\text{const}>0$ and that $\forall t\geq 0$.

We wish to solve the problem of stabilizing the unperturbed motion (1.2) up system (2.1) up to uniform asymptotic stability by the addition of dissipative and gyroscopic forces.

As a consequence of Theorem 1, the following theorem holds.

Theorem 2. *In the case of an even number of coordinates $n = 2k$, the unperturbed motion (1.2) of system (2.1) will be stabilized up to uniform asymptotic stability regardless of the form of the non-linear forces by adding dissipative forces with the diagonal matrix $b(t)\mathbf{E}$ and gyroscopic forces with matrix $\mathbf{G}(t)$, such that*

$$\mathbf{G}(t) = \mu(t)\mathbf{P}(t), \quad \mu(t) \geq \mu_0 = \text{const} > 0$$

$$b(t) \geq b_0 = \text{const} > 0, \quad \dot{\mu}(t) \leq b(t)\mu(t) - \delta, \quad \delta = \text{const} > 1$$

Corollary 2. *In the case of an even number of coordinates $n = 2k$, the unperturbed motion (1.2) of system (2.1) will be stabilized up to uniform asymptotic stability regardless of the form of the non-linear forces by adding dissipative forces with diagonal matrix $b(t)\mathbf{E}$ and gyroscopic forces with matrix $\mathbf{G}(t)$, such that*

$$b(t) \geq b_0 = \text{const} > 0, \quad \mathbf{G}(t) = \mu\mathbf{P}(t), \quad \mu = \text{const} > 0, \quad b_0\mu > 1$$

Remark 2. Theorem 2 and Corollary 2 supplement Theorem 3.2 in Kosov's paper[6] in that no constraints are imposed on the derivative $\dot{\mathbf{P}}(t)$ of the matrix of the non-conservative positional forces and the coefficient $b(t)$ in front of the matrix of the dissipative forces can depend on time.

3. The sufficient conditions for stabilization of the motions of mechanical systems with assigned non-conservative and potential forces

Let the equations of the perturbed motion of a mechanical system have the form

$$\ddot{\mathbf{q}} + (\text{diag}\{c_1(t), \ldots, c_n(t)\} + \mathbf{P}(t))\mathbf{q} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) \quad (3.1)$$

where $\mathbf{q} \in R^n$ is the vector of the generalized coordinates, $\text{diag}\{c_1(t), \ldots, c_n(t)\}$ is the matrix of the potential forces, the skew-symmetric matrix $\mathbf{P}(t) \in R^{n \times n}$ with continuous bounded elements describes the non-conservative forces acting on the system, $\mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ is a function that contains $\mathbf{q}$ and $\dot{\mathbf{q}}$ in an order higher than the first order, and $\mathbf{Q}(t, \mathbf{0}, \mathbf{0}) = \mathbf{0}$.

We wish to solve the problem of stabilizing the unperturbed motion (1.2) of system (3.1) up uniform asymptotic stability by the addition of dissipative and gyroscopic forces.

Theorem 3. *Let the matrix of the potential forces in system (3.1) have the form $c(t)\mathbf{E} \in R^{n \times n}$, and let the continuous bounded function $c(t)$ satisfy the condition $c_1 \leq c(t) \leq c_2$, where $c_1, c_2 = \text{const} > 0$. Then the unperturbed motion (1.2) of system (3.1) will be stabilized up to uniform asymptotic stability if dissipative forces with diagonal matrix $b(t)\mathbf{E}$ ($b(t)$ is a continuous function, and $0 < b_1 \leq b(t) \leq b_2$) and gyroscopic forces with the matrix $\mathbf{G}(t) = \mu\mathbf{P}(t)$ are added to the system and if $b_1 \geq \sqrt{2c_2}$ and the positive coefficient μ satisfies the condition*

$$\mu \in [\mu^-, \mu^+], \quad \mu^{\pm} = \frac{b_1 \pm \sqrt{b_1^2 - 2c_2}}{c_2}$$

Proof. After the dissipative and gyroscopic forces are added, system (3.1) takes the form

$$\mathcal{L}(\mathbf{q}) + c(t)\mathbf{q} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) \quad (3.2)$$

We introduce the notation

$$\tilde{b}(t) = b(t) - 1/\mu, \quad d(t) = b(t) - \mu c(t) - 1/\mu$$

Using a previously proven theorem (Ref. 12, Theorem 2) with the matrix $\mathbf{C} = \text{diag}\{1/\mu\}$, we find that the inequality

$$|d(t)| \leq \tilde{b}(t)$$

ensures uniform asymptotic stability of the zero equilibrium position (1.2) of system (3.2). Hence, taking into account the constraints imposed on the functions $b(t)$ and $c(t)$, we obtain the proof of the theorem.

Remark 3. In Theorem 3 no constraints are imposed on the derivatives of the matrices in Eq. (3.2); therefore, it is impossible to apply exponential stability theorem 2.1 from Kosov's paper[6] to system (3.2).

Theorem 4. *Let the matrix of the potential forces in system (3.1) have the form $c(t)\mathbf{E} \in R^{n \times n}$, where $c(t)$ is a continuous bounded function. The unperturbed motion (1.2) of system (3.1) will be stabilized up to uniform asymptotic stability if the system is given an addition of dissipative accelerating forces with diagonal matrix $b(t)\mathbf{E}$ ($b(t)$ is a continuous bounded function) and gyroscopic forces with matrix*

$$\mathbf{G}(t) = \mu\mathbf{P}(t), \quad \mu = \text{const} > 0,$$

which are such that the following condition holds for any $t_0 \geq 0$ for all $t \geq t_0$

$$\int_{t_0}^{t} \max\left\{-\frac{1}{\mu}, \frac{1}{\mu} - e(t)\right\} dt \leq L = \text{const}; \quad e(t) = b(t) + \frac{\mu\dot{c}(t) - \dot{b}(t)}{2d(t)} \quad (3.3)$$

The constant L does not depend on the choice of t_0.

Proof. After the dissipative accelerating and gyroscopic forces are added, system (3.1) takes the form (3.2). In Eq. (3.2) we make the replacement of variables

$$\mathbf{x}_1 = \mathbf{q}, \quad \mathbf{x}_2 = \beta(t)(\mathbf{q} + \mu\dot{\mathbf{q}})$$

where $\beta(t)$ is a certain continuously differentiable function, to be determined, which satisfies the conditions

$$\beta(t) \geq \beta_0 = \text{const} > 0, \quad \forall t \geq 0$$

Then we obtain the system

$$\dot{\mathbf{x}}_1 = -\frac{1}{\mu}\mathbf{x}_1 + \frac{1}{\mu\beta(t)}\mathbf{x}_2$$

$$\dot{\mathbf{x}}_2 = \beta(t)(d(t)\mathbf{E} + \mathbf{G}(t))\mathbf{x}_1 - \left(\tilde{b}(t)\mathbf{E} + \mathbf{G}(t) - \frac{\dot{\beta}(t)}{\beta(t)}\mathbf{E}\right)\mathbf{x}_2 \tag{3.4}$$

and choose the Lyapunov function for system (3.4) in the form

$$V(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 \tag{3.5}$$

where $|\cdot|$ denotes a spherical vector norm in the space R^n. Then, by virtue of system (3.4), for the derivative $\dot{V}(t, \mathbf{x}_1, \mathbf{x}_2)$ we obtain the estimate

$$\dot{V}(t, \mathbf{x}_1, \mathbf{x}_2) \leq 2\lambda_{\max}(\mathbf{A}(t))V(\mathbf{x}_1, \mathbf{x}_2) - W(t, \mathbf{x}_1, \mathbf{x}_2); \quad W(t, \mathbf{x}_1, \mathbf{x}_2) \geq 0 \tag{3.6}$$

where $\lambda_{\max}(\mathbf{A}(t))$ is the maximum eigenvalue of the matrix

$$\mathbf{A}(t) = \begin{Vmatrix} -\frac{1}{\mu} & \frac{1}{2\mu\beta(t)} + \frac{\beta(t)d(t)}{2} \\ \frac{1}{2\mu\beta(t)} + \frac{\beta(t)dt}{2} & \frac{\dot{\beta}(t)}{\beta(t)} - \tilde{b}(t) \end{Vmatrix}$$

We obtain

$$2\lambda_{\max}(\mathbf{A}(t)) = \frac{\dot{\beta}(t)}{\beta(t)} - b(t) + \sqrt{\left(\frac{2}{\mu} - b(t) + \frac{\dot{\beta}(t)}{\beta(t)}\right)^2 + \left(\frac{1}{\mu\beta(t)} + \beta(t)d(t)\right)^2}$$

We choose the function $\beta(t)$ so that the second term under the radical vanishes identically. Here it must be assumed that the following inequality holds:

$$d(t) \leq \varepsilon = \text{const} < 0, \quad \forall t \geq 0$$

Then we obtain

$$\beta(t) = \frac{1}{\sqrt{-\mu d(t)}}, \quad \forall t \geq 0 \tag{3.7}$$

Therefore,

$$\lambda_{\max}(\mathbf{A}(t)) = \frac{1}{2}\left(-e(t) + \left|\frac{2}{\mu} - e(t)\right|\right)$$

For the stability of the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$ of system (3.4), it is sufficient that a function $a(t)$ exists such that

$$\lambda_{\max}(\mathbf{A}(t)) \leq a(t), \quad \int_{t_0}^{t} a(s)ds \leq L = \text{const}, \quad \forall t \geq t_0$$

If the number L does not depend on t_0, the stability will be uniform.

We will find the function $a(t)$. We will have

$$-2a(t) - e(t) \leq \frac{2}{\mu} - e(t) \leq 2a(t) + e(t)$$

We choose the function $a(t)$ in the form

$$a(t) = \max\left\{-\frac{1}{\mu}, \frac{1}{\mu} - e(t)\right\}$$

Then, for uniform stability of the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$ of system (3.4), it is sufficient that inequality (3.3) should hold. In fact, the comparison equation $\dot{u} = 2a(t)u$ is uniformly stable under condition (3.3).

Using the asymptotic stability theorem from Ref. 11, we can note that condition (3.3) also ensures uniform asymptotic stability of the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$ of system (3.4). In fact, the set

$$\{W^*(t, \mathbf{x}_1, \mathbf{x}_2) = 0\} \tag{3.8}$$

is contained in the set $\{(a^*(t) + 1/\mu)|\mathbf{x}_1| = 0\}$. Hence we find that the solutions of a limit system for (3.4), that are contained in set (3.8), include only the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$. Thus, all the conditions of the previously proved theorem (Ref. 11, Theorem 1) hold.

Remark 4. In condition (3.3) it is assumed that the coefficient $c(t)$ in front of the matrix of the potential forces can take negative values. For example, if a mechanical system with one degree of freedom is described by the equation

$$\ddot{q} + c(t)q = 0 \tag{3.9}$$

with a coefficient $c(t)$ that satisfies the condition that there are constants $\varepsilon > 0$ and L such that the inequality

$$\int_{t_0}^{t} c(t)dt \geq \varepsilon(t - t_0) - L, \quad \forall t \geq t_0$$

holds for any $t_0 \geq 0$, then adding the dissipative accelerating force $f(t)\dot{q}$ with the coefficient

$$f(t) = (c(t) - \varepsilon)/\alpha + \alpha$$

will stabilize the system up to uniform asymptotic stability. Here the number $\alpha > 0$ was selected so that

$$c(t) \leq \varepsilon + \alpha^2, \quad \forall t \geq 0$$

Remark 5. Special cases in which condition (3.3) holds are defined by the following systems of inequalities

$$\mu b(t) \geq 1, \quad d(t) \leq \varepsilon = \text{const} < 0, \quad \mu\dot{c}(t) \leq \dot{b}(t) + 2d(t)\left(b(t) - \frac{2}{\mu}\right) \tag{3.10}$$

$$d(t) \leq \varepsilon = \text{const} < 0, \quad \mu\dot{c}(t) \geq \dot{b}(t) + 2d(t)\left(b(t) - \frac{2}{\mu}\right) \tag{3.11}$$

Remark 6. The unperturbed motion (1.2) of system (3.1) will be stabilized up to uniform asymptotic stability if the system is given an addition of dissipative accelerating forces with diagonal matrix $\mathrm{diag}\{b_1(t), \ldots, b_n(t)\}$ ($b_i(t)$ $i = 1, \ldots, n$) are continuous bounded functions and gyroscopic forces with matrix $\mathbf{G}(t) = \mu\mathbf{P}(t)$ ($\mu = \text{const} > 0$) which are such that inequalities, similar to (3.11) where $c(t)$ and $b(t)$ are replaced by $c_i(t)$ and $b_i(t)$, where $i = 1, \ldots, n$, hold for all $t \geq 0$.

Theorem 5. *Let the coefficients $c_i(t)$ ($i = 1, \ldots, n$) of the matrix of the potential forces in (3.1) be negative, and let the following condition hold*

$$\det(\mathrm{diag}\{c_1(t), \ldots, c_n(t)\} + \mathbf{P}(t)) \geq \varepsilon = \text{const} > 0$$

Then the unperturbed motion (1.2) of system (3.1) will be stabilized up to uniform asymptotic stability if dissipative forces with the diagonal matrix $b(t)\mathbf{E}$ ($b(t)$ is a continuous bounded function) and gyroscopic forces with the matrix

$$\mathbf{G}(t) = \mu(t)\mathbf{P}(t), \quad \mu(t) \geq \mu_0 = \text{const} > 0$$

which are such that

$$2B(t) \geq \max\{\mu(t)c_i(t), \delta\}, \quad i = 1, \ldots, n$$

$$B(t) = b(t) - \frac{1}{\mu(t)} - \frac{\dot{\mu}(t)}{\mu(t)}, \quad \delta = \text{const} > 0 \tag{3.12}$$

are added to the system.

Proof. After the dissipative and gyroscopic forces are added, system (3.1) takes the form

$$\mathscr{L}(\mathbf{q}) + \mathrm{diag}\{c_1(t), \ldots, c_n(t)\}\mathbf{q} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{3.13}$$

As a result of the replacement of variables

$$\mathbf{x}_1 = \mathbf{x}, \quad \mathbf{x}_2 = \mathbf{x} + \mu(t)\dot{\mathbf{x}}$$

we obtain the system

$$\dot{\mathbf{x}}_1 = -\frac{1}{\mu(t)}\mathbf{x}_1 + \frac{1}{\mu(t)}\mathbf{x}_2$$

$$\dot{\mathbf{x}}_2 = (B(t)\mathbf{E} + \mathbf{G}(t) - \mu(t)(\mathrm{diag}\{c_i(t)\} + \mathbf{P}(t)))\mathbf{x}_1 - (B(t)\mathbf{E} + \mathbf{G}(t))\mathbf{x}_2 \tag{3.14}$$

For system (3.14) we take the vector Lyapunov function in the form

$$\mathbf{V} = (V_1, V_2)^T, \quad V_1 = |\mathbf{x}_1|, \quad V_2 = |\mathbf{x}_2|$$

where $|\cdot|$ denotes a Euclidean norm in the space R^n. Then we obtain the reference system

$$\begin{aligned} \dot{u}_1 &= -\frac{1}{\mu(t)}u_1 + \frac{1}{\mu(t)}u_2 \\ \dot{u}_2 &= \max_{i=1,\ldots,n} |\mu(t)c_i(t) - B(t)|u_1 - B(t)u_2 \end{aligned} \tag{3.15}$$

which is uniformly stable. It can be proved by analogy with Theorem 1 that the set $\{\max\{V_1, V_2\} = \text{const} > 0\}$ does not contain solutions of a limit system for (3.14), which ensures uniform asymptotic stability of the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$ of system (3.14).

Remark 7. The matrix of the dissipative forces in Theorem 5 can be chosen in the form $\text{diag}(b_1(t), \ldots, b_n(t))$ with the continuous bounded functions

$$b_i(t), \quad 0 < b_0 \le b_i(t) \le b_{\max}, \quad i = 1, \ldots, n, \quad t \ge 0$$

Then conditions (3.12) should be replaced by the inequalities

$$\frac{2}{\mu(t)} - b_i(t) - b_j(t) + \frac{2\dot{\mu}(t)}{\mu(t)} \le -\mu(t)c_i(t)$$

$$\mu(t)c_i(t) \ge b_i(t) - b_j(t), \quad i, j = 1, \ldots, n$$

Example 2. Consider the problem of stabilizing of the rotational motion of an axisymmetric satellite in an elliptic orbit.

Let $Ox_1y_1z_1$ be an orbital system of coordinates formed by a radius vector, a transversal and a binormal to the orbit. Here $Oxyz$ is a system of coordinates that is rigidly connected to the body, whose axes are the principal central axes of inertia of the satellite, where Oz is the axis of symmetry, and $A = B$ and C are the principal central moments of inertia of the satellite. The transition from the $Oxyz$ system to the $Ox_1y_1z_1$ system is specified by the Euler angles ψ, θ and φ (ψ is the angle of precession, θ is the angle of nutation, and φ is the intrinsic angle of rotation). If the true anomaly is taken as the independent variable, the equations of motion of the satellite will have the form[14,15]

$$\begin{aligned} &\zeta^2\psi''\sin\theta - 2e\zeta\sin\nu\psi'\sin\theta + 2\zeta^2\psi'\theta'\cos\theta - 2\zeta^2\theta'\sin\theta\cos\psi - \alpha\beta\theta' - \\ &- \zeta^2\sin\theta\sin\psi\cos\psi - \alpha\beta\sin\psi - 2e\sin\nu\zeta\cos\theta\cos\psi = 0 \\ &\zeta^2\theta'' - 2e\sin\nu\zeta\theta' + 2\zeta^2\psi'\sin^2\theta\cos\psi - \zeta^2(\psi')^2\sin\theta\cos\theta + \alpha\beta\psi'\sin\theta + \\ &+ (\zeta\cos^2\psi + 3(1-\alpha))\zeta\sin\theta\cos\theta + \alpha\beta\cos\theta\cos\psi - 2e\zeta\sin\nu\sin\psi = 0 \end{aligned} \tag{3.16}$$

where

$$\zeta = 1 + e\cos\nu, \quad \beta = \frac{r_0}{\omega_0}, \quad \alpha = \frac{C}{A}\ (0 \le \alpha \le 2), \quad \omega_0 = \frac{R_0}{p}\sqrt{\frac{g_0}{p}}$$

$$r_0 = \omega_0(1 + e\cos\nu)^2(\psi'\cos\theta + \varphi' - \cos\psi\sin\theta) = \text{const}$$

R_0 is the equatorial radius of the earth, g_0 is the acceleration due to gravity at the equator, p is a parameter of the orbit, ν is the true anomaly, e is the eccentricity of the orbit, and the prime denotes differentiation with respect to ν.

Consider the problem of stabilization using the dissipative moments of the generalized steady motion

$$\theta = \pi/2, \quad \psi = \pi \tag{3.17}$$

under which the satellite rotates about the axis of symmetry Oz, which is collinear to a normal to the orbital plane, with angular velocity

$$\varphi' = r_0/(\omega_0\zeta^2) - 1$$

We select the control moments in the form

$$M_x = -\bar{k}_1 p_a - \bar{k}_2 p_r, \quad M_y = -\bar{k}_1 q_a - \bar{k}_2 q_r, \quad M_z = 0 \tag{3.18}$$

where p_a and q_a are the projections of the absolute angular velocity onto the Ox and Oy axes, p_r and q_r are the projections of the relative angular velocity of the satellite onto these axes, and $\bar{k}_1$ and $\bar{k}_2$ are constant positive coefficients. The control moments (3.18) correspond to the generalized forces

$$L_\psi = -(\bar{k}_1 + \bar{k}_2)\omega\psi'\sin^2\theta - \bar{k}_1\omega\sin\theta\cos\theta\cos\psi$$

$$L_\theta = -(\bar{k}_1 + \bar{k}_2)\omega\theta' - \bar{k}_1\omega\sin\psi$$

$$L_\varphi = 0$$

Here $\omega = d\nu/dt = \omega_0\zeta^2$. Thus, we will solve the problem of stabilizing of the rotation of a satellite by the addition of dissipative and non-conservative positional forces.

Realization of the moments (3.18) is possible when sensors of the absolute and relative angular velocities with sensitivity axes directed along Ox and Oy and a device that imparts moments that are proportional to the signals of these sensors are used.

We introduce the notation

$$k_i = \bar{k}_i/(A\omega_0), \quad i = 1, 2$$

Then the equations in variations take the form

$$\mathbf{q}'' + (\operatorname{diag}\{b(\nu) + k_2/\zeta^2\} + \mathbf{G}(\nu))\mathbf{q}' + (\operatorname{diag}\{c_1(\nu), c_2(\nu)\} + \mathbf{P}(\nu))\mathbf{q} = 0 \tag{3.19}$$

where

$$b(\nu) = \frac{k_1 - 2e\zeta\sin\nu}{\zeta^2}, \quad c_1(\nu) = \frac{\alpha\beta}{\zeta^2} - 1, \quad c_2(\nu) = c_1(\nu) - \frac{3(1-\alpha)}{\zeta}$$

$$\mathbf{G}(\nu) = \begin{Vmatrix} 0 & 1 - c_1(\nu) \\ -1 + c_1(\nu) & 0 \end{Vmatrix}, \quad \mathbf{P}(\nu) = \begin{Vmatrix} 0 & b(\nu) \\ -b(\nu) & 0 \end{Vmatrix}$$

We introduce the function

$$\mu(\nu) = \frac{2\zeta^2 - \alpha\beta}{k_1 - 2e\zeta\sin\nu}$$

Then, by Theorem 5, the stabilization conditions take the form

$$\mu(\nu) \geq \mu_0 = \text{const} > 0, \quad c_i(\nu) \geq c_0 = \text{const} > 0$$

$$2(1 - \mu(\nu)(b(\nu) + k_2/\zeta^2) + \mu'(\nu)) \leq -\mu^2(\nu)c_i(\nu), \quad i = 1, 2 \tag{3.20}$$

Note that the problem of stabilizing the relative equilibrium of the satellite by adding dissipative and non-conservative positional forces has been solved by several investigators (see, for example, Refs 14, 16 and 6), but the case of a circular orbit was considered. Thus, condition (3.20) for stabilizing the rotational motion of a satellite in an elliptic orbit supplements the results in those papers.

Example 3. Consider the problem of the stability of a non-tumbling gyro horizon consisting of four identical gyroscopes, mounted on a fixed platform and joined in pairs using an antiparallelogram so that the members of each pair turn about the axes of the frames in opposite directions through angles of equal magnitude. The linearized equations of the perturbed motion have the form[7]

$$\tilde{J}_2\ddot{\alpha} + 2H\dot{\delta} + b_1\dot{\alpha} + Mga\alpha + f\delta = 0$$

$$\tilde{J}_1\ddot{\beta} - 2H\dot{\gamma} + b_1\dot{\beta} + Mga\beta - f\gamma = 0$$

$$\tilde{J}_0\ddot{\gamma} + 2H\dot{\beta} + b_2\dot{\gamma} + \kappa_1\gamma + f\beta = 0$$

$$\tilde{J}_0\ddot{\delta} - 2H\dot{\alpha} + b_2\dot{\delta} + \kappa_2\delta - f\alpha = 0$$

$$\tilde{J}_i = J_i + 2(A_c + A + B_c), \quad i = 1, 2, \quad \tilde{J}_0 = 2(A_c + A) \tag{3.21}$$

where the angles α and β characterize the deviation of the platform from the plane of the horizon, the angles γ and δ characterize the deviation of the axes of rotation of the gyroscopes, H is the angular momentum of the gyroscopes, J_1 and J_2 are the moments of inertia of the platform about the axes of its suspension, A_c and B_c are the equatorial and polar moments of inertia of each gyroscope frame, A is the equatorial moment of inertia of each gyroscope rotor, b_1 and b_2 are the dissipation coefficients on the axes of the suspensions of the platform and the gyroscope frames, respectively, Mg is the weight of the platform, a is the distance from the centre of gravity of the platform to the points of support, κ_1 and κ_2 are the shiftness of the springs that join the central links of the antiparallelogram to the platform, and $f > 0$ is the coefficient of proportionality of the signals supplied by the sensors to the axes of the platform and the gyroscope frames.

Using Theorem 5, we obtain the condition for asymptotic stability of the zero solution of system (3.21)

$$f(b_1 - k_0\tilde{J}_i) \le 2HMga \le f\left(b_1 + \left(k_0 - \frac{f}{H}\right)\tilde{J}_i\right)$$
$$f(b_2 - k_0\tilde{J}_0) \le 2H\kappa_j \le f\left(b_2 + \left(k_0 - \frac{f}{H}\right)\tilde{J}_0\right); \quad i, j = 1, 2 \tag{3.22}$$

where

$$k_0 = \min\left\{\frac{b_1}{\tilde{J}_2}, \frac{b_1}{\tilde{J}_1}, \frac{b_2}{\tilde{J}_0}\right\}$$

A similar problem was considered previously in Ref. 7 under the assumption that the following inequalities hold

$$J_1 > J_2, \quad \kappa_1 > \kappa_2, \quad \tilde{J}_0 Mga > \tilde{J}_1\kappa_1$$

Thus, condition (3.22) supplements the result obtained in Ref. 7.

Now consider the case in which the coefficient of proportionality f is a continuously differentiable bounded function of time that satisfies the conditions

$$0 < f_{\min} \le f(t) \le f_{\max}, \quad \forall t \ge 0$$

Then, applying Theorem 5, we obtain the following condition for asymptotic stability of the zero solution of system (3.21)

$$\dot{f}(t) \ge \dot{f}_{\min} = \text{const}$$
$$f_m(b_1 - k_0\tilde{J}_i) \le 2HMga \le f_m\left(b_1 + \left(k_0 - \frac{f_m}{H} + \frac{2\dot{f}_{\min}}{f_m}\right)\tilde{J}_i\right)$$
$$f_m(b_2 - k_0\tilde{J}_0) \le 2HM\kappa_j \le f_m\left(b_2 + \left(k_0 - \frac{f_m}{H} + \frac{2\dot{f}_{\min}}{f_m}\right)\tilde{J}_0\right)$$
$$i, j = 1, 2; \quad m = \min, \max$$

Example 4. Consider the problem of the stable functioning of a gyrovertical with radial correction.

The equations of motion of the axis of a gyrovertical have the form[13,17]

$$A\ddot{\alpha} + b\dot{\alpha} - H\dot{\beta} - k\beta = X_1$$
$$A\ddot{\beta} + b\dot{\beta} + H\dot{\alpha} + k\alpha = X_2 \tag{3.23}$$

where A is the equatorial moment of inertia of the gyroscope, H is the angular momentum, b is the coefficient of the resistance forces, k is the slope of the characteristic graph of the moment sensors, and X_1 and X_2 are non-linear functions of α, β, $\dot{\alpha}$, $\dot{\beta}$.

Let the coefficient k in (3.23) be a continuously differentiable bounded function of time that satisfies the conditions

$$0 < k_{\min} \le k(t) \le k_{\max}, \quad \forall t \ge 0$$

Using Theorem 5, we obtain the following condition for uniform asymptotic stability of the zero solution $\alpha = \beta = \dot{\alpha} = \dot{\beta} = 0$:

$$\dot{k}(t) \ge \dot{k}_{\min} > -\min\left\{k_{\min}\left(\frac{b}{A} - \frac{k_{\min}}{H}\right), k_{\max}\left(\frac{b}{A} - \frac{k_{\max}}{H}\right)\right\} \tag{3.24}$$

which is identical with the necessary and sufficient condition obtained using the Routh–Hurwitz criterion in the case when $k = \text{const} > 0$.

Note that such a problem was previously considered in Ref. 6, where the following condition for exponential stability was obtained

$$0 < k_{\min} \le k(t) \le k_{\max} < \frac{bH}{A}, \quad |\dot{k}(t)| \le \dot{k}_{\max} \le 2A\left(\frac{bH}{Ak_{\max}} - 1\right)\frac{k_{\min}^2}{H} \tag{3.25}$$

It is notable that the first inequality in condition (3.25) follows from inequality (3.24). Unlike the second inequality in (3.25), no upper limit is imposed on the derivative $\dot{k}(t)$ in condition (3.24).

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